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Calculation and application of complex function integration

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ABSTRACT
Complex integration is an important topic in the theory of complex functions, primarily focusing on analytic functions and proving their essential properties through complex integration. The computation of complex integrals, particularly the application of Cauchy's Integral Theorem, is one of the key problems in integral theory. Complex functions have broad applications in both theory and practice, and analytic functions play a crucial role in solving practical problems in fields such as physics. Formula, the Newton-Leibniz Formula, the higher-order derivative formulas for analytic functions, and the Residue Theorem. These methods are invaluable in solving complex integrals. This paper will first introduce the basic concepts and properties of complex functions, then systematically summarize several common methods for calculating complex integrals, illustrating their internal connections through examples. By highlighting some tips for solving complex integrals, this paper aims to help efficiently select the appropriate method in practical problems, improving problem-solving speed and effectiveness.

KEYWORDS
complex function integration; Cauchy integration formula; Newton Leibniz formula residue theorem

1 Introduction

Complex variable function is a basic course in mathematics. Its main research object is analytic function. Analytic function has a wide range of applications in theory and practice. The integral theory of complex variable function is an important part of the theory of complex variable function. The application of the integral theory of complex variable function is one of the important tools for studying analytic function. Many important properties of analytic function can be proved by using complex integral. For example, to prove that "the derivative of analytic function is continuous" and "the derivatives of analytic function exist", which are propositions that seem to be only related to differential calculus, complex integral is generally used[1]. In the integral theory of complex variable function, Cauchy's integral theorem and Cauchy's integral formula are particularly important. They are the basic theorem and basic formula of complex variable function theory. Cauchy's integral theorem is the theoretical basis of the integral of analytic function. The Cauchy's integral formula derived from the Cauchy's integral theorem shows that an analytic function in a region can be expressed by an integral, that is, the value of the analytic function at any point in the region can be expressed by its value on the boundary through integration. This shows that there is a close connection between the values of analytic function. The Cauchy's integral formula can also be used to prove the formula of high-order derivatives of analytic function. An analytic function has derivatives of any order, so the derivative of an analytic function is still an analytic function. It can be said that the differential calculus of complex functions is based on integral calculus, which is completely different from the situation of real functions. Therefore, the operation of complex integrals has a very important position and significance. In February 2003, Professor Zhang Kunshi of the Higher Correspondence College studied "Residue Theorem and Integration of Complex Functions", which discussed the internal connection between the residue theorem and the integration of complex functions, and gave examples to illustrate the close relationship between the residue theorem, Cauchy integral theorem, Cauchy integral formula and Cauchy high-order derivative formula. In March 2006, Cui Dongling of Huainan Normal University studied "Calculation Methods of Complex Integrals", in which he revealed the internal connection of many methods through variable substitution, Cauchy integral formula, Cauchy integral theorem and residue theorem. They have made many progresses in studying the calculation methods of complex integrals and proved the calculation methods of complex function integrals. The calculation methods of complex function integrals are flexible and diverse. This article will use the basic principles of complex function integration and several basic methods of complex integrals, give examples for each calculation method, and calculate complex integrals through the Cauchy integral theorem, Cauchy integral formula, Cauchy higher-order derivative formula, residue theorem, etc., thereby revealing the internal connection between many methods, making a relatively systematic summary of the calculation methods of complex integrals, and summarizing the problem-solving methods and techniques for complex function integrals[2].

The calculation of complex integrals can be divided into open paths and contour paths according to the integral path. The calculation result of the integral may be related to the integral path or may be unrelated. The integral path of the integrand may be analytical or may have singular points. For various complex integral calculation methods, we can choose parametric equation method, Newton-Leibniz formula, Cauchy integral theorem, Cauchy integral formula, high-order derivative formula, residue theorem, series method, logarithmic residue argument theorem, singular point method, Laplace transform method. In the calculation of complex integrals, we can choose the appropriate method to solve the problem according to the characteristics of the integrand and the integral path[3].

2 Theoretical knowledge

Lemma 1Directed line segment: Let C be a smooth (or piecewise smooth) curve on a given plane, and then choose one of the two possible directions of C as the square (or positive direction), then we call C a curve with direction, or simply a directed curve[4].

If the positive direction of curve C is from A to B, recorded as C_0. Then the negative direction of curve C is from B to A, recorded as C_0^-. Generally, the positive direction of curve C always refers to the direction from the starting point to the end point. On the contrary, the negative direction of curve C is the direction from the end point to the starting point, recorded as C_0^-. The definition of the direction of the closed curve C is the counterclockwise direction, that is, the positive direction, recorded as C. The clockwise direction is the negative direction, recorded as C_0^-. For the boundary line C of the region, the positive direction of C means that when point P on the curve moves along this direction, the area enclosed by C is always to the left of point P. The positive direction of the boundary line C of the simply connected region is counterclockwise: recorded as C_0^-. The positive direction of the outer boundary line C of the multi-connected region is counterclockwise and recorded as C_0^-. The positive direction of the inner boundary line is clockwise and recorded as C_1^-.

Lemma 2Definition of complex integral

Suppose that on the complex plane, is a smooth curve l: $y=y(x)$ (with continuous derivatives) with as the starting point z_0 and as the end point $\dot{z}$. Take a series of points $z_0,z_1,\cdots,z_{n-1},z_n=\dot{z}$ on to divide l into segments. Take any point ξ_k on each small segment $z_{k-1}z_k$ and take the sum $s_n=\sum_{k=1}^{n} f(\xi_k)(z_k-z_{k-1})=\sum_{k=1}^{n} f(\xi_k)\Delta z_k\ \Delta z_k=z_k-z_{k-1}$. , as $.n\to\infty$, and the length of each small segment tends to zero, if $\lim_{n\to\infty} S_n$ exists, then it is called $f(z)$ is integrable along l, and $\lim_{n\to\infty} S_n$ is called the integral of $f(z)$ along path l, denoted by $\int_l f(z)dz$.[5]

$$\int_l f(z)dz=\lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\Delta z_k$$

($f(z)$ takes value on l, that is, z changes on l).

3 Several common methods for calculating complex integrals

3.1 Definition method to calculate complex integrals

Definitionof complex integral: Let the function $w=f(z)$ be defined in region D, and let C be a smooth directed curve in D with starting point A and end point B[6]. Divide the curve C into n arc segments arbitrarily, and let the dividing points be

$$A=z_0,z_1,z_2,\cdots,z_{k-1},z_k,\cdots,z_n=B$$

Pick any point ξ_k on the arc segment z_{k-1} to z_k $(k=1,2,\cdots,n)$ and construct the sum:

$$s_n=\sum_{k=1}^{n} f(\xi_k)(z_k-z_{k-1})=\sum_{k=1}^{n} f(\xi_k)\Delta z_k,$$

Define $\Delta z_k=z_k-z_{k-1}$, and the length of the arc segment as $\delta=\max\left\{\Delta S_k\right\}(k=1,2,\cdots,n)$, when $\delta\to 0$, no matter how C is divided, that is whichever ξ_k is picked, there is a unique limit s_n, then the limit value is called the integral of the function $f(z)$ along the curve C.

$$\int_c f(z)dz=\lim_{\delta\to\infty}\sum_{k=1}^{n} f(\xi_k)\Delta z_k$$

Example 3.2.1 Calculate the complex integral: $\int_c \left(x+iy\right)^2 d_z$, where C is the positive (counterclockwise) circle with radius 1 and the origin as the center.

Step 1: Parameterize the integration path

Choose an appropriate parameterization. For a circle centered at the origin, a common parameterization is: $z\left(t\right)=e^{it}$, where t ranges from 0 to 2π. For this parameterization, we obtain $d_z=ie^{it}d_t$

Step 2: Replace $z\left(t\right)$ and d_z with values above, we obtain:

$$\int_c \left(x+iy\right)^2 d_z=\int_0^{2\pi}\left(e^{it}\right)^2\cdot ie^{it}dt=\int_0^{2\pi} ie^{3it}\,dt$$

Step 3: Calculate the integral. Next, we calculate the above integral:

$$\int_0^{2\pi} ie^{3it}\,dt=i\int_0^{2\pi}\left(cos\left(3t\right)+isin\left(3t\right)\right)dt$$

This integral can be calculated in two parts: real part and imaginary part:

Real part: $\int_0^{2\pi} \cos(3t)\, dt = 0 \cos(3t)$ (because the positive and negative parts are symmetrical within a period).

Imaginary part: $\int_0^{2\pi} \sin(3t)\, dt = 0 \sin(3t)$. (Similarly, because the positive and negative parts are symmetrical within a period.)

Therefore, the result of the integration is: $\int_0^{2\pi} ie^{3it}\, dt = i \cdot (0 + i \cdot 0) = 0$

The conclusion $\int_0^{2\pi} ie^{3it}\, dt = i \cdot (0 + i \cdot 0) = 0$ is obtained by definition. This result shows that the complex integral of the function $(x+iy)^2$ is zero on a given integral path. This calculation process not only demonstrates the calculation steps of complex integrals, but also reflects the powerful application of complex variable function theory in analyzing the integrals of specific types of functions.

3.2 Newton-Leibniz formula for calculating complex integrals

Lemma 3: If $f(z)$ is an analytic function in region D, $\phi(z)$ is the antiderivative of $f(z)$. z_1,z_2 are two points in D, then

$$\int_{z_1}^{z_2} f(z)\, dz = \phi(z_2) - \phi(z_1) \quad (3.2.1)$$

Note that when using the Newton-Leibniz formula to calculate complex integrals, we must pay attention to the simply connected region D. The integral value has nothing to do with the specific path, but only with the starting point and the end point. Our original function is an elementary function[7].

3.3 Calculating complex integrals using the Cauchy integral formula

Lemma 4Cauchy integral formula: Let the boundary of region D be a contour, the function $f(z)$ is analytic in region D, C is any positive simple closed curve in D and its interior is completely contained in D, for any point z_0 in C, if $f(z) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0}$ then

$$\oint_c \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0). \quad (3.3.1)$$

3.4 Converting complex function integrals into real function curve integrals of real variables

Assume that the complex function $f(z)$ is defined on region D, C is a calculable length curve (or a piecewise smooth curve) on $\dot{D}$, and assume that $\int_c f(z)dz$ exists. Assume that $f(z) = u(x,y) + iv(x,y)$ is continuous along curve C, then

$$\int_c f(z)dz = \int_c udx - vdy + i \int_c udy + vdx \quad (3.4.1)$$

According to the characteristics of the parameter equation of curve C, formula (3.4.1) can be transformed into the following three specific calculation formulas:

Form 1: When the parametric equations of the smooth curve are $x = x(t)$ and $y = y(t)\left(\alpha \le t \le \beta, \text{or } \beta \le t \le \alpha\right)$ and $t = \alpha,\beta$, which correspond to the starting point and end point of C respectively, equation (3.4.1) can be transformed into

$$\int_c f(z)dz = \int_\alpha^\beta \left[u(x(t),y(t))x'(t) - v(x(t),y(t))\right]dt$$
$$+ i\int_\alpha^\beta \left[u(x(t),y(t))y'(t) + v(x(t),y(t))x'(t)\right]dt \quad (3.4.2)$$

Form 2: When the equation of the smooth curve C is $y = y(x)\ a \le x \le b$, then equation (3.4.1) can be transformed into

$$\int_c f(z)dz = \int_a^b \left[u(x,y(x)) - v(x,y(x))y'(x)\right]dx$$
$$+ i\int_a^b \left[u(x,y(x))y'(x) + v(x,y(x))\right]dx \quad (3.4.3)$$

Form 3: When the smooth curve C is $z = z(t)\left(\alpha \le t \le \beta, or\ \beta \le t \le \alpha\right)$, $t = \alpha,\beta$, corresponds to the starting point and end point of C respectively[8]. Then formula (3.4.1) can be expressed as

$$\int_c f(z)dz = \int_\alpha^\beta f(z(t))z'(t)dt \quad (3.4.4)$$

When the integral path C is a complex integral of an open curve, there are 2 situations as follows: when the integral is related to the path, the parametric equation of the curve C is used to calculate the complex integral. when the integral is independent of the path, the original function method is used to solve it. When the integral path C is a closed curve, the Cauchy integral theorem and corollary are used for the complex integral[9].

3.5 Cauchy's theorem and its corollary to calculate complex integrals

Lemma 5Cauchy integral theorem: Let the function $f(z)$ be analytic in a simply connected region D on the complex plane, and let C be any contour in D, then $\int_c f(z)dz = 0$. (Simply connected region.)

The equivalent form of the Cauchy Integral Theorem: Let C be a contour, C be its interior, and $f(z)$ is analytic it in a closed domain, then $\overline{D} = D + C$.

$$\int_c f(z)dz = 0 \quad (3.5.1)$$

Lemma 6If D is a multiply connected region, we have the following theorem: Let D be a connected region consisting of complex contours $c = c_0 + c_1 + c_2 + \cdots + c_n$, $f(z)$ is analytic in D, continuous on $\overline{D} = D + C$, then

$$\int_c f(z)dz = 0 \quad (3.5.2)$$

Note: When applying the corollary of the Cauchy integral theorem to calculate the integral, the following points should be noted: First, pay attention to the different directions of the curves; second, pay attention to the fact that it must be a simple closed curve; third, pay attention to the fact that several internal curves do not contain each other and do not intersect each other; fourth, all closed curves constitute the boundary of the region[10]. If and only if these conditions are fully met, the corollary of the Cauchy integral theorem can be applied to transform the loop integral along the outer boundary line of the region into the integral along the inner boundary curve of the region.

Composite closed circuit theorem: If $f(z)$ is analytic in the region enclosed by the composite closed circuit C $\left(= C_0 + C_1^- + C_2^- + \cdots + C_n^-\right)$ is completely contained in D in the multiply connected region D, then $\oint_c f(z)dz = 0$, that is, $\oint_c f(z)dz = \sum_{k=1}^{n} \oint_{c_k} f(z)dz$.

3.6 Calculate complex integrals using higher-order derivative formulas of analytical functions

Lemma 7Higher-order Derivative Formula: If function $f(z)$ is analytic in region D, C is any positive simple closed curve in D, whose interior is completely contained in D, and z_0 is any point in C, then

$$\oint_c \frac{f(z)}{\left(z - z_0\right)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}\left(z_0\right) n = 1,2,\cdots \quad (3.6.1)$$

3.7 Residue Theorem for Complex Integrals

Lemma 8Residue Theorem: If the function $f(z)$ is analytic everywhere in the region except for a finite number of isolated singular points $m_1, m_2, \cdots, m_n m_1, m_2, \cdots, m_n$, and C is continuous except for any positive simple closed curve in D that contains each singular point, then

$$\oint_c f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Re} s\left[f(z), z_k\right] \quad (3.7.1)$$

Let m be the pole of order n of $f(z)$, $f(z) = \frac{\varphi(z)}{(z-m)^n}$, where $\varphi(z)$ is analyzed at point m, $\varphi(m) \neq 0$, then $\operatorname{Re} sf(z) = \frac{\varphi^{(n-1)}(a)}{(n-1)!}$, where the symbol $\varphi^{(0)}(m)$ represents $\varphi(m)$, and we have $\varphi^{(n-1)}(m) = \lim_{z \to m} \varphi^{(n-1)}(z)$.

3.8 Other methods for calculating complex integrals

Use the result to evaluate the complex integral: $\oint_c \frac{1}{\left(z - z_n\right)} dz$

(1) When z_0 is inside C,

$$\oint_c \frac{1}{(z-a)^{n+1}}dz=\begin{cases}2\pi i,n=0\\0,n\neq 0\end{cases} \quad (3.8.1)$$

(2) When z_0 is outside C,

$$\int_c \frac{1}{(z-z_0)^n}dz=0 \quad (3.8.2)$$

Lemma 9Logarithmic residue argument principle: Logarithmic residue theorem: If $f(z)$ is analytic and non-zero on a simple curve C, and is analytic everywhere in the interior of C except for a finite number of poles, then $\frac{1}{2\pi i}\oint_c \frac{f'(z)}{f(z)}dz=N-P$ [11]. Where N is the total number of zeros in C. When calculating the number of zeros and poles of $f(z)$, zeros or poles of level m are counted as m zeros or poles. Argument principle: If is analytic on a simple closed curve C and in C, and is non-zero on C [12], then the number of zeros of $f(z)$ in C is equal to $\frac{1}{2\pi}$ multiplied by the change in argument of $f(z)$ when z travels around C in the positive direction, that is

$$N=\frac{1}{2\pi}\Delta_c+Argf(z)$$

Lemma 10Series method: Continuity term-by-term integration theorem: If $f_n(z)(n=1,2,3,\cdots)$. $\sum_{n=1}^{+\infty}f_n(z)$ is continuous on the curve C and converges uniformly to $f(z)$ on C, then is continuous on $f(z)$. The curve C and can be integrated term-by-term along C:

$$\int_c \sum_{n=1}^{+\infty}f_n(z)\,dz=\int_c f(z)\,dz$$

Lemma 11: Singularity mining method: According to the fact that the integral along the outer boundary is equal to the sum of the integrals along the inner boundary, the singular points in the region contained in the integral path are contained in a closed curve using several mutually exclusive closed curves (and these closed curves are all contained in C), and then the complex integral is calculated using the Cauchy integral formula or the high-order derivative formula[13].

Lemma 12Laplace transform method[14]: Let $f(t)$ be a real-valued function or complex-valued function defined on $[0,+\infty]$. If the integral $\int_0^{+\infty} f(t)e^{-pt}dt$ involving a complex variable $P=\sigma+is(\sigma,s\in R)$ exists in some region of P, then the complex function defined by this integral $F(p)=\int_0^{+\infty} f(t)e^{-pt}dt$ is called the Laplace transform of the function $.f(t)$, denoted by $F(p)=L(f(t))$.

4 Summary

This article mainly summarizes the calculation and application of complex integrals. Through examples and theorem deductions, in the calculation part, it summarizes various complex integral calculation problems and solution methods. The calculation method of complex variable integrals is relatively flexible. The choice of method is critical when solving problems. You can decide which calculation method to use based on whether the integral path is a closed curve, the specific form of the integrand, and the analyticity in the given area. Each method has its own advantages and disadvantages. In the process of solving problems, choose the appropriate method to solve the problem according to the characteristics of the question type. Each method complements each other and there is an inevitable connection. There are still many difficult problems that have not been solved in the complex number domain, and many problems have not been overcome. The several methods for calculating complex variable function integrals summarized in this article hope to provide convenience for researchers.

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